Limits and colimits in various categories of institutions

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CALCO-jnr
20 August 2007, Bergen, Norway

Short agenda

1. An introduction to institutions
2. Limits and colimits of diagrams of institutions
3. Relating limits in different categories of institutions

Part I

An introduction to institutions

Definition

- An institution is a formalisation of a logical system
- It consist of:
  1. a category of signatures $\mathbf{Sign}$
  2. a model functor $\mathbf{Mod}\colon \mathbf{Sign}^{op} \to \mathbf{Cat}$
  3. a sentence functor $\mathbf{Sen}\colon \mathbf{Sign} \to \mathbf{Set}$
  4. for each signature $\Sigma \in |\mathbf{Sign}|$ a satisfaction relation $\models_\Sigma$

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  4. for each signature $\Sigma \in |\mathbf{Sign}|$ a satisfaction relation $\models_\Sigma$
- For each signature morphism $\sigma\colon \Sigma \to \Sigma'$, the satisfaction condition must hold:

$$\mathbf{Mod}(\sigma)(m) \models_\Sigma \phi \iff m \models_{\Sigma'} \mathbf{Sen}(\sigma)(\phi)$$

Institution — example — equational logic ($\mathbf{EQ}$)

Example

- Signatures: **many-sorted algebraic signatures**
- Sentences for a signature Σ: **equations** of the form $\forall X.t = t'$ (t and t' are terms with variables from the set X over the signature Σ)
- Models for a signature Σ: **algebras** with homomorphisms
- The satisfaction relation is defined in the usual way

Example

Having a signature morphism $\sigma : \Sigma \to \Sigma'$ we can:

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- for $\phi \in \mathbf{Sen}(\Sigma)$, build a sentence $\sigma(\phi) \in \mathbf{Sen}(\Sigma')$, with all symbols ”translated”

Institution — example — equational logic ($\mathbf{EQ}$)

Example

Having a signature morphism $\sigma : \Sigma \to \Sigma'$ we can:

- for $\phi \in \mathbf{Sen}(\Sigma)$, build a sentence $\sigma(\phi) \in \mathbf{Sen}(\Sigma')$, with all symbols "translated"
- for a model $m' \in |\mathbf{Mod}(\Sigma')|$, build its reduct $m'|_\sigma \in |\mathbf{Mod}(\Sigma)|$

How to connect institutions?

Definition

A morphism between institutions $\mathbf{I}$ and $\mathbf{I}'$ consists of:

- a functor $\Phi : \mathbf{Sign} \to \mathbf{Sign}'$
- a natural transformation $\beta : \mathbf{Mod} \to \Phi^{op};\mathbf{Mod}'$
- a natural transformation $\alpha : \Phi;\mathbf{Sen}' \to \mathbf{Sen}$

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Such that the following satisfaction condition holds:

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A morphism shows, how a richer institution is built over a simpler one.

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Definition

A comorphism between institutions $\mathbf{I}$ and $\mathbf{I}'$ consists of:

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Comorphisms show, how a simpler institution can be represented in a more complex one.

Morphisms and comorphisms — example

Having institutions **EQ** and **FOEQ** we can build:

Example

- a morphism from **FOEQ** to **EQ**
  - when translating signatures, we forget about the relations
  - first-order models are also models for equational logic
  - an equation is also a first-order-with-equality formula

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Example

- a comorphism from **EQ** to **FOEQ**
  - when translating signatures, we add an empty (multi-sorted) set of relational symbols
  - again: first-order models are also models for equational logic
  - again: an equation is also a first-order formula

Part II

Limits and colimits of diagrams of institutions

Diagrams of institutions

- Using morphisms or comorphisms, we can build two categories of institutions: **INS** and **coINS**

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- And build diagrams of institutions, in one of those categories

Diagrams of institutions

- Using morphisms or comorphisms, we can build two categories of institutions: $\mathbf{INS}$ and $\mathbf{coINS}$
- And build diagrams of institutions, in one of those categories

Note

$\mathbf{INS} \neq (\mathbf{coINS})^{op}$

Where do these diagrams come from?

- Such diagrams appear in heterogeneous specifications
- Various parts of the system are specified in various specification languages
- Each specification is built on top of an institution
- Specifications are linked by morphisms or comorphisms

Completeness

Theorem (Tarlecki, 1986)

$\mathbf{INS}$ *is complete.*

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Also, similarly:

Theorem (Also known for a long time)

$\mathbf{coINS}$ *is complete.*

Completeness

Theorem (Tarlecki, 1986)

INS *is complete.*

Also, similarly:

Theorem (Also known for a long time)

coINS *is complete.*

- Limits are easy to construct (in a component-by-component manner)
- The construction of categories of models and sentence sets is independent of the structure of the signature category

Equalizers in $\mathbf{INS}$

$$\mathbf{I} \xrightarrow{\langle \Phi, \beta, \alpha \rangle} \mathbf{I}_1 \underset{\langle \Phi_2, \beta_2, \alpha_2 \rangle}{\overset{\langle \Phi_1, \beta_1, \alpha_1 \rangle}{\rightrightarrows}} \mathbf{I}_2$$

Construction

- $\mathbf{Sign}$ and Φ: equalizer of Φ_1 i Φ_2

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- For a given signature $\Sigma \in |\mathbf{Sign}|$, model category and $\beta(\Sigma)$: equalizer of $\beta_1(\Sigma)$ and $\beta_2(\Sigma)$

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- For a given signature $\Sigma \in |\mathbf{Sign}|$, model category and $\beta(\Sigma)$: equalizer of $\beta_1(\Sigma)$ and $\beta_2(\Sigma)$
- For a given signature $\Sigma \in |\mathbf{Sign}|$, sentence set and $\alpha(\Sigma)$: coequalizer of $\alpha_1(\Sigma)$ and $\alpha_2(\Sigma)$

Cocompleteness

Fact

INS *and* **coINS** *are not cocomplete.*

Cocompleteness

Fact

$\mathbf{INS}$ and $\mathbf{coINS}$ are not cocomplete.

However due to purely set-theoretical issues; we can get rid of those problems, by considering institutions, whose signature categories are small ($\mathbf{sINS}$ and $\mathbf{scoINS}$). Then we get:

Cocompleteness

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The embeddings $\mathbf{sINS} \hookrightarrow \mathbf{INS}$ and $\mathbf{scoINS} \hookrightarrow \mathbf{coINS}$ are cocontinuous.

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But the constructions are much uglier:

- coproducts — easy
- coequalizers — not so easy

Coequalizers in $\mathbf{sINS}$

$$\mathbf{I}_1 \underset{\langle \Phi_2, \beta_2, \alpha_2 \rangle}{\overset{\langle \Phi_1, \beta_1, \alpha_1 \rangle}{\rightrightarrows}} \mathbf{I}_2 \xrightarrow{\langle \Phi, \beta, \alpha \rangle} \mathbf{I}$$

Construction

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Coequalizers in sINS

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Construction

- $\mathbf{Sign}$ and Φ: coequalizer of Φ_1 i Φ_2
- For a given signature $\Sigma \in |\mathbf{Sign}|$, model category $\mathbf{Mod}(\Sigma)$: colimit of diagram, which has vertices:
  - all model categories in $\mathbf{I}_2$ for signatures Σ_2 such that there exists a morphism in $\mathbf{Sign}$ from Σ to $\Phi(\Sigma_2)$
  - all model categories in $\mathbf{I}_1$ for signatures Σ_1 such that there exists a morphism in $\mathbf{Sign}$ from Σ to $\Phi(\Phi_1(\Sigma_1))$
  - and some morphisms $(\ldots)$

Coequalizers in $\mathbf{sINS}$

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  - and some morphisms $(\ldots)$
- Similarly, $\mathbf{Sen}(\Sigma)$ is a limit of a (diagram as above)^{op}.

Observation

Only when the categories of signatures are small, we can be sure that we will be always able to define a diagram as described above.

Colimits in $\mathbf{sINS}$

Observation

Only when the categories of signatures are small, we can be sure that we will be always able to define a diagram as described above.

Observation

Intuitive duality between morphisms and comorphisms fails when considering limits and colimits.

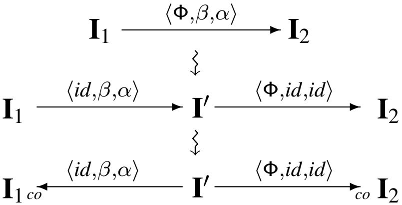
Part III

Relating limits in different categories of institutions

The need to relate morphisms and comorphisms

- It may be easier to work with morphisms
- It may be easier to work with comorphisms
- Diagrams of institutions may be “inconsistent” — include morphisms and comorphisms

Changing morphisms into a span of comorphisms



$$\mathbf{I}' = \langle \mathbf{Sign}_1, \Phi^{op};\mathbf{Mod}_2, \Phi;\mathbf{Sen}_2, \Phi;\models_2 \rangle$$

Changing morphisms into a span of comorphisms

$$\mathbf{I}_1 \xrightarrow{\langle \Phi, \beta, \alpha \rangle} \mathbf{I}_2$$

$$\rightsquigarrow$$

$$\mathbf{I}_1 \xrightarrow{\langle id, \beta, \alpha \rangle} \mathbf{I}' \xrightarrow{\langle \Phi, id, id \rangle} \mathbf{I}_2$$

$$\rightsquigarrow$$

$$\mathbf{I}_1 \; {}_{co}\!\xleftarrow{\langle id, \beta, \alpha \rangle} \mathbf{I}' \xrightarrow{\langle \Phi, id, id \rangle}_{co} \; \mathbf{I}_2$$

$$\mathbf{I}' = \langle \mathbf{Sign}_1, \Phi^{op};\mathbf{Mod}_2, \Phi;\mathbf{Sen}_2, \Phi;\models_2 \rangle$$

Why are spans good?

When the comorphisms are treated as relations and combined, they express the same relation as the original morphism.

Limits of diagrams built of spans

- If we change all morphisms into spans in a diagram, its shape changes
- How to construct the limit of diagram of spans, knowing that comorphisms have a certain shape?
- How does this limit correspond to a limit of the original diagram?

"Flattening" a diagram

Let **Sign** be a limit of a diagram, which is a projection of a diagram of institutions on the categories of signatures.

Flattening

We can alter the diagram of institutions, and change each vertex so that its signature category is **Sign**.

$$\langle \mathbf{Sign}_i, \mathbf{Mod}_i, \mathbf{Sen}_i, \models_i \rangle \rightsquigarrow \langle \mathbf{Sign}, \Phi_i^{op};\mathbf{Mod}_i, \Phi_i;\mathbf{Sen}_i, \Phi_i;\models_i \rangle$$

Similarly with morphisms; each morphism will have $id_{\mathbf{Sign}}$ on the signature coordinate.

“Flattening” a diagram

$\mathbf{INS}_{\mathbf{Sign}}$

We have created a diagram in the category $\mathbf{INS}_{\mathbf{Sign}}$ — of institutions with a fixed signature category.

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Morphisms and comorphisms

$\langle id, \beta, \alpha \rangle : \mathbf{I}_1 \to \mathbf{I}_2$ is an institution morphism if and only if $\langle id, \beta, \alpha \rangle : \mathbf{I}_2 \to_{co} \mathbf{I}_1$ is an institution comorphism.

"Flattening" a diagram

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We have created a diagram in the category $\mathbf{INS_{Sign}}$ — of institutions with a fixed signature category.

Morphisms and comorphisms

$\langle id, \beta, \alpha \rangle : \mathbf{I}_1 \rightarrow \mathbf{I}_2$ is an institution morphism if and only if $\langle id, \beta, \alpha \rangle : \mathbf{I}_2 \rightarrow_{co} \mathbf{I}_1$ is an institution comorphism.

Corollary

$\mathbf{INS_{Sign}} \cong (\mathbf{coINS_{Sign}})^{op}$.

What does “flattening” a diagram give us?

Translations:

- limit of the flattened diagram $\Rightarrow$ limit of the original diagram
- co-limit of the flattened diagram $\Rightarrow$ limit of a diagram of spans
- (co-limit of the flattened diagram $==$ limit of the flattened co-diagram)

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Observation

Easy construction of limits of diagrams of spans without much additional trouble.

Relating limits in $\mathbf{INS}$ and limits of spans in $\mathbf{coINS}$

Suppose that:

- $\mathbf{I}$ is the limit of a diagram $\mathbf{D}$ in $\mathbf{INS}$
- $\mathbf{I}''$ is the limit of a diagram of spans of $\mathbf{D}$ in $\mathbf{coINS}$

Relating limits in $\mathbf{INS}$ and limits of spans in $\mathbf{coINS}$

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Intuitively,

- $\mathbf{I}$ is the “most complex” institution
- $\mathbf{I}''$ is the “simplest” institution
- institution morphism shows how a more complex institution is built over a simpler one

Hence, there may be a morphism from $\mathbf{I}$ to $\mathbf{I}''$.

Relating limits in $\mathbf{INS}$ and limits of spans in $\mathbf{coINS}$

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Hence, there may be a morphism from $\mathbf{I}$ to $\mathbf{I}''$.

Fact

Using flattened diagrams, it is easy to build one morphism for each component of the graph.

Thank you

Questions?