Limits and colimits in categories of institutions

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1 Introduction

The theory of institutions, first introduced by Goguen and Burstall in 1984 ([GB83,GB92]), quickly gained ground and proved to be a very useful tool to construct and reason about logics in a uniform way. Since then, it has found many applications and has been widely developed. An institution is a formalization of a logical system—for example, we can build an institution of equational logic or first order logic.

There are two main ways of moving between institutions, using either institution morphisms or comorphisms. Informally, morphisms express how a "richer" institution is built over a "simpler" one; comorphisms express a relation going the other way round: how a "simpler" institution can be encoded in a "richer" one. These intuitions hint at some duality between the two concepts. Various properties of (co)morphisms are presented very thoroughly and systematically in [GR02]. Taking morphisms or comorphism, we can build two categories: **INS** and **coINS**, with institutions as objects.

In this article, I am going to analyse some of the relationships between limits and colimits of diagrams built from institutions linked by morphisms and comorphisms, as well as show the constructions of those limits and colimits. Even though morphisms and comorphisms may seem to be dual concepts at first, universal constructions associated with morphisms and comorphisms turn out to be rather different.

The main motivation behind this work takes source in heterogeneous specifications [Mos02b,Tar00], which are built over a number of institutions linked with morphisms or comorphisms. It is sometimes important to have the underlying diagram of institutions represented in a uniform way, using only morphisms or only comorphisms; hence the need to translate one into another. One way to do that is by transforming a morphism into a span of (co)morphisms (or vice versa), as introduced for example in [Mos02b]. Also, given such a diagram, it may be useful to represent a family of models of a heterogeneous distributed specification, or specifications themselves in an institution, which combines the institutions involved. Limits/colimits of institutions haven't proved to be the best tool for "putting institutions together" (see for example [GB85,Paw95]), however it may be suitable to use them as concise representations of institutions diagrams. This approach is different from the one taken by, for example, Mossakowski ([Mos02a,Mos06]) and Diaconescu ([Dia02]), where, given a diagram, the corresponding Grothendieck institution is built. Using this technique, institutions are put into one, essentially "side by side", without much interaction. We are after a more compact representation, where some combination of signatures, models and sentences of the institutions involved takes place. One of the constructions to consider, when pursuing such a goal, is the construction of limits and colimits of diagrams of institutions.

Now comes the question: how do (co)limits of diagrams relate to (co)limits of diagrams built by replacing each morphism by a span of comorphisms? (Or each comorphism by a span of morphisms.) Here, we will show what the answer is for the case of limits. When morphisms are replaced by spans of comorphisms, the shape of the diagram changes. Hence the "procedure" for constructing limits changes as well; even though the new morphisms are of a special form. In general it seems there is no simple and straightforward way to translate between limits/colimits of the two diagrams, which shows that morphisms and comorphisms are not entirely dual.

All proofs of correctness of constructions and of theorems are left out. They are available in [War07].

2 Definitions

This section presents definitions used later in the article: of institutions, institution morphisms and comorphisms and various institution categories. Examples of these concepts can be found in [GB83], [GR02], [ST88] and many other papers dealing with institutions.

Definition 1. *An* institution $\mathbf{I} = \langle \mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models \rangle$ *consists of:*

- *a category* $\mathbf{Sign}$ *of signatures,*
- *a functor* $\mathbf{Mod}\colon \mathbf{Sign}^{op} \to \mathbf{Cat}$*, which assigns to each signature a category of models.* $\mathbf{Cat}$ *is a category of "all" categories and functors between them,*
- *a functor* $\mathbf{Sen}\colon \mathbf{Sign} \to \mathbf{Set}$*, which assign to each signature a set of sentences,*
- *for each signature* $\Sigma \in |\mathbf{Sign}|$*, a satisfaction relation* $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times \mathbf{Sen}(\Sigma)$*,*

Such that for each signature morphism $\sigma\colon \Sigma \to \Sigma'$*, sentence* $\varphi \in \mathbf{Sen}(\Sigma)$ *and model* $m' \in |\mathbf{Mod}(\Sigma')|$*, the satisfaction condition holds (*$\mathbf{SC}$*):*

$$m' \models_{\Sigma'} \mathbf{Sen}(\sigma)(\varphi) \Longleftrightarrow \mathbf{Mod}(\sigma)(m') \models_\Sigma \varphi.$$

The following notations are used: $\sigma(\varphi)$ stands for $\mathbf{Sen}(\sigma)(\varphi)$ and $m'|_\sigma$ stands for $\mathbf{Mod}(\sigma)(m')$.

The satisfaction condition takes then the form:

$$m' \models_{\Sigma'} \sigma(\varphi) \Longleftrightarrow m'|_\sigma \models_\Sigma \varphi.$$

Definition 2. *An* institution morphism $\mu\colon \mathbf{I} \to \mathbf{I}'$*,* $\mu = \langle \Phi, \alpha, \beta \rangle$*, where* $\mathbf{I} = \langle \mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models \rangle$ *and* $\mathbf{I}' = \langle \mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models' \rangle$ *consists of:*

- *a functor between signature categories* $\Phi\colon \mathbf{Sign} \to \mathbf{Sign}'$
- *a natural transformation between model functors* $\alpha\colon \mathbf{Mod} \to (\Phi^{op});\mathbf{Mod}'$
- *a natural transformation between sentence functors* $\beta\colon \Phi;\mathbf{Sen}' \to \mathbf{Sen}$.

Also here, the satisfaction condition must hold, for each signature $\Sigma \in |\mathbf{Sign}|$*, sentence* $\varphi' \in \mathbf{Sen}'(\Phi(\Sigma))$*, and model* $m \in |\mathbf{Mod}(\Sigma)|$*:*

$$m \models_\Sigma \beta_\Sigma(\varphi') \Longleftrightarrow \alpha_\Sigma(m) \models'_{\Phi(\Sigma)} \varphi'.$$

Note that the domain of the sentence functor is a "re-indexed" sentence functor of the institution $\mathbf{I}'$, and the codomain is the sentence functor of $\mathbf{I}$.

Intuitively, the institution $\mathbf{I}$ is more complicated than the institution $\mathbf{I}'$. A morphism between them shows how $\mathbf{I}$ is built upon $\mathbf{I}'$.

Definition 3. *An* institution comorphism $\rho\colon \mathbf{I} \to_{co} \mathbf{I}'$, $\rho = \langle \Phi, \alpha, \beta \rangle$*, where we have* $\mathbf{I} = \langle \mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models \rangle$ *and* $\mathbf{I}' = \langle \mathbf{Sign}', \mathbf{Mod}', \mathbf{Sen}', \models' \rangle$ *consists of:*

- *a functor* $\Phi\colon \mathbf{Sign} \to \mathbf{Sign}'$
- *a natural transformation* $\alpha\colon (\Phi^{op});\mathbf{Mod}' \to \mathbf{Mod}$
- *a natural transformation* $\beta\colon \mathbf{Sen} \to \Phi;\mathbf{Sen}'$

such that for each signature $\Sigma \in |\mathbf{Sign}|$*, sentence* $\varphi \in \mathbf{Sen}(\Sigma)$ *and model* $m' \in |\mathbf{Mod}'(\Phi(\Sigma))|$ *the satisfaction condition holds:*

$$m' \models'_{\Phi(\Sigma)} \beta_\Sigma(\varphi) \Longleftrightarrow \alpha_\Sigma(m') \models_\Sigma \varphi.$$

Intuitively, ρ is a representation of institutions—it shows, how a simpler institution can be embedded into a richer one. Institution comorphisms were first introduced under the name "simple maps of institutions" by Meseguer, and as "representations" by Tarlecki in [Tar95].

Definition 4. *Having institutions and morphisms between them, we can build a* category of institutions $\mathbf{INS}$.

- ***Objects:*** *institutions*
- ***Morphisms:*** *institution morphisms as defined in Def. 2.*
- ***Identities:*** *morphisms* $id = \langle id_{\mathbf{Sign}}, id_{\mathbf{Mod}}, id_{\mathbf{Sen}} \rangle$.
- ***Composition:*** *a composition of a morphism* $\mu_1 : \mathbf{I} \to \mathbf{I}'$ *with a morphism* $\mu_2\colon \mathbf{I}' \to \mathbf{I}''$ *is a morphism* $\mu = \mu_1;\mu_2\colon \mathbf{I} \to \mathbf{I}''$*, where for* $\mu_1 = \langle \Phi_1, \alpha_1, \beta_1 \rangle$, $\mu_2 = \langle \Phi_2, \alpha_2, \beta_2 \rangle$ *we define* $\mu = \langle \Phi, \alpha, \beta \rangle$*:*
  $\Phi = \Phi_1;\Phi_2 \quad : \mathbf{Sign} \to \mathbf{Sign}''$
  $\alpha = \alpha_1;((\Phi_1^{op}) \cdot \alpha_2) : \mathbf{Mod} \to (\Phi_1;\Phi_2)^{op};\mathbf{Mod}''$
  $\beta = (\Phi_1 \cdot \beta_2);\beta_1 \quad : \Phi_1;\Phi_2;\mathbf{Sen}'' \to \mathbf{Sen}$

Here $\cdot$ is the horizontal composition of natural transformations, and ; is the composition of functors or the vertical composition of natural transformations (depending on context). It is easy to check that the definition of identities is correct, that composition is associative, and that μ is indeed an institution morphism.

Definition 5. *Using comorphisms instead of morphisms we can also build another* category of institutions, **coINS**.

- ***Objects:*** *institutions*
- ***Morphisms:*** *comorphisms of institutions, as defined in Def. 3.*
- ***Identities:*** *comorphisms* $id = \langle id_{\mathbf{Sign}}, id_{\mathbf{Mod}}, id_{\mathbf{Sen}} \rangle$.
- ***Composition:*** *composition of a comorphism* $\rho_1 \colon \mathbf{I} \to \mathbf{I}'$ *with a comorphism* $\rho_2 \colon \mathbf{I}' \to \mathbf{I}''$ *is a comorphism* $\rho = \rho_1;\rho_2 \colon \mathbf{I} \to \mathbf{I}''$, *where for* $\rho_1 = \langle \Phi_1, \alpha_1, \beta_1 \rangle$, $\rho_2 = \langle \Phi_2, \alpha_2, \beta_2 \rangle$ *we define* $\rho = \langle \Phi_1;\Phi_2, \beta_1;(\Phi_1 \cdot \beta_2), ((\Phi_1^{op}) \cdot \alpha_2);\alpha_1 \rangle$.

Again it is easy to check that **coINS** is a category.

Definition 6. Categories of institutions **sINS** and **scoINS** *are full subcategories of, respectively,* **INS** *and* **coINS**, *where objects are only those institutions, in which signature categories are small (objects and morphisms of the signature category form a proper set).*

Definition 7. Categories of institutions $\mathbf{INS}_{\mathbf{Sign}}$ and $\mathbf{coINS}_{\mathbf{Sign}}$ (with a fixed signature category), where $\mathbf{Sign} \in |\mathbf{Cat}|$ is an arbitrary category *are subcategories of, respectively,* **INS** *and* **coINS**, *where objects are all institutions with a fixed signature category* **Sign**, *and morphisms are all institution morphisms/comorphisms, in which the functor between signature categories is an identity.*

Definition 8. *The* signature-projecting functor $\mathbf{C} \colon \mathbf{INS} \to \mathbf{Cat}$ *is defined as follows*

- $\mathbf{C}(\mathbf{I}) = \mathbf{Sign}$, *for each institution* $\mathbf{I} = \langle \mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models \rangle$
- $\mathbf{C}(\mu) = \Phi$, *for each institution morphism* $\mu : \mathbf{I} \to \mathbf{I}' \in \mathbf{INS}$, *where* $\mu = \langle \Phi, \alpha, \beta \rangle$.

This is a functor which projects an institution on its signature category. We can also define an analogous functor with domain **coINS**.

3 Limits in INS

As mentioned in the introduction, diagrams of institutions often appear in heterogeneous specifications [Mos02b,Tar00]. One way of compactly representing such diagrams is by considering their limits.

Theorem 9. *The category* **INS** *is complete.*

This result is well-known, and the proof can be found for example in [Tar85]. However, the construction given there proceeds rather indirectly in several quite involved steps. Instead, here we give an explicit construction directly in terms of institutions and their morphisms in the diagram, thus offering a better "feel" and direct handle on the result. Here we will describe the construction of arbitrary limits; to do that it is enough to construct products of an arbitrary family of categories and equalizers of any two morphisms ([Mac71, Ch. V]). The constructions are easy and are done in a component-wise manner; the construction of model categories and sentence sets on each signature doesn't depend on the overall structure of the signature category.

3.1 Products in INS

For a given family of institutions, $\mathbf{I}_j \in |\mathbf{INS}|$, $j \in J$, where J is a set of indices and $\mathbf{I}_j = \langle \mathbf{Sign}_j, \mathbf{Mod}_j, \mathbf{Sen}_j, \models_j \rangle$, we define a product of this family, an institution $\mathbf{I} = \Pi_{j\in J}\mathbf{I}_j$.

- $\mathbf{Sign} = \Pi_{j\in J}\mathbf{Sign}_j$ is a product of categories:
  - objects are functions $\xi\colon J \to \biguplus_{j\in J} |\mathbf{Sign}_j|$, such that $\xi(j) \in |\mathbf{Sign}_j|$ for $j \in J$.
  - morphisms between ξ and ξ' are functions $\chi : J \to \biguplus_{j\in J} \mathbf{Sign}_j$, such that $\chi(j)\colon \xi(j) \to \xi'(j)$ in $\mathbf{Sign}_j$.
- $\mathbf{Mod}(\xi) = \Pi_{j\in J}\mathbf{Mod}_j(\xi(j))$ for $\xi \in |\mathbf{Sign}|$ (product of categories)
- $\mathbf{Mod}(\chi) = \chi^{mod}$, where the functor $\chi^{mod}\colon \mathbf{Mod}(\xi') \to \mathbf{Mod}(\xi)$ is defined as follows: $\chi^{mod}(m')(j) = \mathbf{Mod}_j(\chi(j))(m'(j))$, for $\chi\colon \xi \to \xi'$ in $\mathbf{Sign}$, $j \in J$ and $m' \in |\mathbf{Mod}(\xi')|$ (analogically for model morphisms).
- $\mathbf{Sen}(\xi) = \biguplus_{j\in J} \mathbf{Sen}_j(\xi(j))$ (coproduct of sets, its elements are pairs $\langle \varphi, j\rangle$, where $j \in J$ and $\varphi \in \mathbf{Sen}_j(\xi(j))$).
- $\mathbf{Sen}(\chi) = \chi^{sen}$, where $\chi^{sen} : \mathbf{Sen}(\xi) \to \mathbf{Sen}(\xi')$ is defined as follows: $\chi^{sen}(\langle \varphi, j\rangle) = \langle \mathbf{Sen}_j(\chi(j))(\varphi), j\rangle$, for $\chi : \xi \to \xi'$ in $\mathbf{Sign}$, $j \in J$ and $\varphi \in \mathbf{Sen}_j(\xi(j))$.
- satisfaction relation $\models_\xi$ for $\xi \in |\mathbf{Sign}|$, $m \in |\mathbf{Mod}(\xi)|$, $j \in J$ and $\varphi \in \mathbf{Sen}_j(\xi(j))$: $m \models_\xi \langle \varphi, j\rangle \iff m(j) \models^j_{\xi(j)} \varphi$

The projections $\pi_j : \mathbf{I} \to \mathbf{I}_j$ for $j \in J$ are defined in a straightforward way, $\pi_j = \langle \Phi_j, \alpha_j, \beta_j\rangle$:

- $\Phi_j(\xi) = \xi(j)$
- $\alpha_j(\xi)(m) = m(j)$, for $m \in |\mathbf{Mod}(\xi)|$ and similarly for model morphisms
- $\beta_j(\xi)(\varphi) = \langle \varphi, j\rangle$, for $\varphi \in \mathbf{Sen}_j(\xi(j))$

Lemma 10. *α and β are natural transformations and π_j for $j \in J$ are institution morphisms.*

Lemma 11. *$\Pi_{j\in J}\mathbf{I}_j$ with projections π_j for $j \in J$ is a product of institutions $\mathbf{I}_j$ for $j \in J$.*

3.2 Equalizers in INS

Given two "parallel" institution morphisms μ_1, $\mu_2 : \mathbf{I}_1 \to \mathbf{I}_2$, we define their equalizer $\mu\colon \mathbf{I} \to \mathbf{I}_1$, with domain $\mathbf{I} = \langle \mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models\rangle$.

$\mathbf{Sign}$ is the subcategory of $\mathbf{Sign}_1$ such that $\Sigma \in |\mathbf{Sign}| \iff \Phi_1(\Sigma) = \Phi_2(\Sigma)$ and $\sigma \in \mathbf{Sign}(\Sigma, \Sigma') \iff \Phi_1(\sigma) = \Phi_2(\sigma)$. The functor $\Phi\colon \mathbf{Sign} \to \mathbf{Sign}_1$ is the inclusion. Hence, Φ is an equalizer of $\Phi_1, \Phi_2\colon \mathbf{Sign}_1 \to \mathbf{Sign}_2$ in $\mathbf{Cat}$.

For $\Sigma \in |\mathbf{Sign}|$, $\mathbf{Mod}(\Sigma)$ is the subcategory of $\mathbf{Mod}_1(\Phi(\Sigma)) = \mathbf{Mod}_1(\Sigma)$, such that

$$m \in |\mathbf{Mod}(\Sigma)| \iff \alpha_1(\Sigma)(m) = \alpha_2(\Sigma)(m)$$
$$h \in \mathbf{Mod}(\Sigma)(m, m') \iff \alpha_1(\Sigma)(h) = \alpha_2(\Sigma)(h).$$

For $\sigma\colon \Sigma \to \Sigma'$, $\mathbf{Mod}(\sigma) = \mathbf{Mod}_1(\sigma)|_{\mathbf{Mod}(\Sigma)}$ (functor domain restriction).

For $\Sigma \in |\mathbf{Sign}|$, let $\mathbf{Sen}(\Sigma) = \mathbf{Sen}_1(\Phi(\Sigma))/{\equiv_\Sigma} = \mathbf{Sen}_1(\Sigma)/{\equiv_\Sigma}$, where $\equiv_\Sigma$ is the smallest equivalence relation such that $\beta_1(\Sigma)(\varphi) \ \equiv_\Sigma \ \beta_2(\Sigma)(\varphi)$ for all $\varphi \in \mathbf{Sen}_2(\Phi_1(\Sigma))$. The full relation satisfies this condition, and an intersection of two relations satisfying this condition also satisfies it, hence a smallest relation exists.

For $\sigma\colon \Sigma \to \Sigma'$ and $[\psi]_{\equiv_\Sigma} \in \mathbf{Sen}(\Sigma)$, we define:

$$\mathbf{Sen}(\sigma)([\psi]_{\equiv_\Sigma}) = [\mathbf{Sen}_1(\Phi(\sigma))(\psi)]_{\equiv_{\Sigma'}} = [\mathbf{Sen}_1(\sigma)(\psi)]_{\equiv_{\Sigma'}}.$$

Remark 12. The above definition is correct, that is, it does not depend on the choice of ψ.

Hence $\mathbf{Mod}(\Sigma)$ with inclusion $\alpha(\Sigma)\colon \mathbf{Mod}(\Sigma) \to \mathbf{Mod}_1(\Sigma)$ is an equalizer of functors $\alpha_1(\Sigma)$ and $\alpha_2(\Sigma)$ between categories $\mathbf{Mod}_1(\Sigma)$ and $\mathbf{Mod}_2(\Phi_1(\Sigma))$ (the choice of Φ_1 or Φ_2 is not important, as $\Phi_1(\Sigma) = \Phi_2(\Sigma)$ from the construction of the signature category), and $\beta(\Sigma)$ is a coequalizer of $\beta_1(\Sigma), \beta_2(\Sigma)\colon \mathbf{Sen}_2(\Phi_1(\Sigma)) \to \mathbf{Sen}_1(\Sigma)$.

The satisfaction relation for $\Sigma \in |\mathbf{Sign}|$ is defined as follows:

$$m \models_\Sigma [\psi]_{\equiv_\Sigma} \iff \text{ for each } \psi' \in [\psi]_{\equiv_\Sigma}, m \models^1_\Sigma \psi'.$$

Lemma 13. *A morphism $\mu\colon \mathbf{I} \to \mathbf{I}_1$ defined as: $\mu = \langle \Phi, \alpha, \beta \rangle$, where $\beta(\Sigma) = [-]_{\equiv_\Sigma}$ for $\Sigma \in |\mathbf{Sign}|$ is an institution morphism.*

Lemma 14. *μ is an equalizer of μ_1 and μ_2.*

4 Colimits in INS

Another way of combining institutions in a diagram is taking its colimit. Dually to limits, to construct a colimit it suffices to show the construction of coproducts and coequalizers (see [Mac71, Chap. V]). However, colimits of arbitrary diagrams of institutions connected by morphisms do not always exist, because it is not always possible to construct a coequalizer of two morphisms. A counter example can be found in [GR02, Ex. 4.10].

However, the problems are purely set-theoretical. If we restrict our attention only to institutions, in which signature categories are small (in typical examples it is enough to restrict the alphabet of symbols used to build operation names), we will get the following result.

Theorem 15. *The category* **sINS** *is cocomplete.*

This result is also not new, and is mentioned for example in [GR02] and proved in [Ros99]. However again, no direct and explicit constructions are given there.

Below the constructions of coproducts and coequalizers are briefly described. It is relatively easy to construct coproducts (the construction is dual to the construction of products in **INS**) but the construction of coequalizers is much harder. Here, as opposed to limits, the constructions of model and sentence functors heavily depend on the overall structure of the signature category.

4.1 Coproducts in INS

For a given family of institutions $\mathbf{I}_j$, $j \in J$, where J is a set of indices, we define its coproduct, an institution $\mathbf{I} = \uplus_{j\in J}\mathbf{I}_j$.

- $\mathbf{Sign} = \biguplus_{j\in J} \mathbf{Sign}_j$ is a coproduct of categories:
  - objects are pairs $\langle \Sigma, j\rangle$, where $j \in J$ and $\Sigma \in |\mathbf{Sign}_j|$.
  - morphisms are pairs $\langle \sigma, j\rangle \colon \langle \Sigma, j\rangle \to \langle \Sigma', j\rangle$, where $j \in J$ and $\sigma \colon \Sigma \to \Sigma'$ is a morphism in $\mathbf{Sign}_j$; for $j \neq j'$, there are no morphisms between $\langle \Sigma, j\rangle$ and $\langle \Sigma', j'\rangle$.
- for $\langle \Sigma, j\rangle \in |\mathbf{Sign}|$, $\mathbf{Mod}(\langle \Sigma, j\rangle) = \mathbf{Mod}_j(\Sigma)$, $\mathbf{Sen}(\langle \Sigma, j\rangle) = \mathbf{Sen}_j(\Sigma)$
- for $\langle \sigma, j\rangle \in \mathbf{Sign}$, $\mathbf{Mod}(\langle \sigma, j\rangle) = \mathbf{Mod}_j(\sigma)$, $\mathbf{Sen}(\langle \sigma, j\rangle) = \mathbf{Sen}_j(\sigma)$
- satisfaction relation: for a signature $\langle \Sigma, j\rangle$, model $m \in |\mathbf{Mod}(\langle \Sigma, j\rangle)| = |\mathbf{Mod}_j(\Sigma)|$ and sentence $\varphi \in \mathbf{Sen}(\langle \Sigma, j\rangle) = \mathbf{Sen}_j(\Sigma)$, $m \models_{\langle \Sigma, j\rangle} \varphi \iff m \models^j_\Sigma \varphi$

The inclusions $\iota_j \colon \mathbf{I}_j \to \mathbf{I}$, $\iota_j = \langle \Phi_j, \alpha_j, \beta_j\rangle$ for $j \in J$, are defined as follows: $\Phi_j(\Sigma) = \langle \Sigma, j\rangle$, $\alpha_j(\Sigma) = id_{\mathbf{Mod}_j(\Sigma)}$ and $\beta_j(\Sigma) = id_{\mathbf{Sen}_j(\Sigma)}$.

Lemma 16. *α and β are natural transformations and ι_j for $j \in J$ are institution morphisms.*

Lemma 17. $\mathbf{I}$ *with inclusions ι_j for $j \in J$ is a coproduct of institutions $\mathbf{I}_j$, for $j \in J$.*

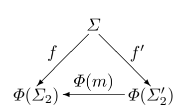
4.2 Coequalizers in sINS

Given two "parallel" morphisms μ_1, $\mu_2 \colon \mathbf{I}_1 \to \mathbf{I}_2$ ($\mathbf{I}_1, \mathbf{I}_2 \in |\mathbf{sINS}|$, we will define their coequalizer $\mu \colon \mathbf{I}_2 \to \mathbf{I}$. The following construction is inspired by [TBG91, Ch. 3, Ex. 4], and coincides with the construction of a left Kan extension in a category of functors with a fixed codomain ([Mac71, Ch. X], [Ros99]).

$\mathbf{Sign}$ is the domain of a coequalizer of functors Φ_1, $\Phi_2 \colon \mathbf{Sign}_1 \to \mathbf{Sign}_2$. The construction of coequalizers in $\mathbf{Cat}$ can be found in [MB99]. It is a bit more complicated then in $\mathbf{Set}$, but they are roughly analogous. Objects in $\mathbf{Sign}$ are equivalence classes of the smallest equivalence relation $\equiv\ \subseteq |\mathbf{Sign}_2| \times |\mathbf{Sign}_2|$ such that for all $\Sigma \in |\mathbf{Sign}_1|$, $\Phi_1(\Sigma) \equiv \Phi_2(\Sigma)$. Morphisms can be defined in a similar way.

Let $\Sigma \in |\mathbf{Sign}|$ be an arbitrary signature. We define a graph $\mathbf{G}_\Sigma$ as follows:

- **nodes**:
  - $\langle \Sigma_1, f, 1\rangle$, where $\Sigma_1 \in |\mathbf{Sign}_1|$, $f : \Sigma \to \Phi(\Phi_1(\Sigma_1))$ in $\mathbf{Sign}$ (choosing Φ_1 or Φ_2 does not matter, because from the construction of Φ we have $\Phi_1;\Phi = \Phi_2;\Phi$).
  - $\langle \Sigma_2, f, 2\rangle$, where $\Sigma_2 \in |\mathbf{Sign}_2|$, $f \colon \Sigma \to \Phi(\Sigma_2)$ in $\mathbf{Sign}$
- **edges**:



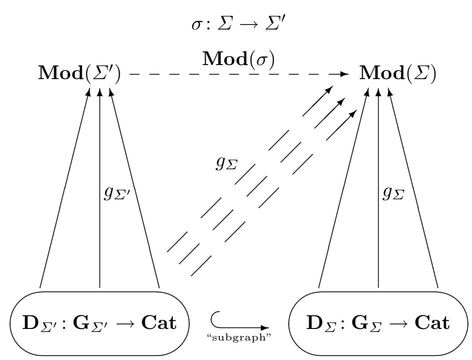
- $m\colon \langle \Sigma_2, f, 2\rangle \to \langle \Sigma_2', f', 2\rangle$, where $m\colon \Sigma_2' \to \Sigma_2$ in $\mathbf{Sign}_2$, is such that $f';\Phi(m) = f$.
- $\langle n_i, m\rangle\colon \langle \Sigma_1, f, 1\rangle \to \langle \Sigma_2', f', 2\rangle$, $i = 1, 2$, $m\colon \Sigma_2' \to \Phi_i(\Sigma_1)$ in $\mathbf{Sign}_2$, is such that $f';\Phi(m) = f$. □

Informally, all of the above nodes are needed so that we can define the model functor on signature morphisms. The first type of edges ("m") is needed to ensure that the resulting construction will be universal; and finally the "$\langle n_i, m\rangle$" edges are there so that the construction will have the coequalizer property.

Remark 18. *Note that only when the category* **Sign** *is small, we can be sure that we will be able to define the graph* $\mathbf{G}_\Sigma$ *(with a* set *of nodes and edges). This is provided by the fact that when both* $\mathbf{Sign}_1$ *and* $\mathbf{Sign}_2$ *are small categories,* **Sign** *is also a small category.*

Next, we define a diagram $\mathbf{D}_\Sigma\colon \mathbf{G}_\Sigma \to \mathbf{Cat}$ as follows:

– $\mathbf{D}_\Sigma(\langle \Sigma_2, f, 2\rangle) = \mathbf{Mod}_2(\Sigma_2)$
– $\mathbf{D}_\Sigma(\langle \Sigma_1, f, 1\rangle) = \mathbf{Mod}_1(\Sigma_1)$
– $\mathbf{D}_\Sigma(m) = \mathbf{Mod_2}(m)$
– $\mathbf{D}_\Sigma(\langle n_i, m\rangle) = \alpha_i(\Sigma_1);\mathbf{Mod}_2(m)$, where $\langle n_i, m\rangle\colon \langle \Sigma_1, f, 1\rangle \to \langle \Sigma_2', f', 2\rangle$. □



Let $\mathbf{Mod}(\Sigma)$ be a colimit of the diagram $\mathbf{D}_\Sigma$ in the category **Cat**. The injection (functor) of $\mathbf{D}_\Sigma(\langle \Sigma_i, f, i\rangle)$ into the colimit $\mathbf{Mod}(\Sigma)$ we will denote by $g_\Sigma^{\langle \Sigma_i, f, i\rangle}\colon \mathbf{Mod}_i(\Sigma_i) \to \mathbf{Mod}(\Sigma)$, $i = 1, 2$.

Let $\sigma : \Sigma \to \Sigma'$ be an arbitrary morphism in **Sign**. We define **Mod** on this morphism. Firstly, we build a cocone for the diagram $\mathbf{D}_{\Sigma'}$, with a vertex $\mathbf{Mod}(\Sigma)$. Injections into this cocone will be denoted by $k_{\Sigma'}^{\langle \Sigma_i', f, i\rangle}\colon \mathbf{Mod}_i(\Sigma_i') \to \mathbf{Mod}(\Sigma)$. The graph $\mathbf{G}_{\Sigma'}$ is a "subgraph" of $\mathbf{G}_\Sigma$: each node of the form $\langle \Sigma_i', f, i\rangle$ in $\mathbf{G}_{\Sigma'}$ has a corresponding node $\langle \Sigma_i', \sigma;f, i\rangle$ in $\mathbf{G}_\Sigma$; moreover, the values of the two nodes and of any edges between corresponding nodes (in $\mathbf{G}_{\Sigma'}$) are identical in diagrams $\mathbf{D}_\Sigma$ and $\mathbf{D}_{\Sigma'}$. Hence, if for an injection into the cocone's vertex from the value of a node $\langle \Sigma_i', f, i\rangle$ we take $k_{\Sigma'}^{\langle \Sigma_i', f, i\rangle} = g_\Sigma^{\langle \Sigma_i', \sigma;f, i\rangle}$, we will get a cocone over $\mathbf{D}_{\Sigma'}$ with a vertex in $\mathbf{Mod}(\Sigma)$. Let $\mathbf{Mod}(\sigma)\colon \mathbf{Mod}(\Sigma') \to \mathbf{Mod}(\Sigma)$

be the unique morphism (which exists, as $\mathbf{Mod}(\Sigma')$ is a colimit of the diagram $\mathbf{D}_{\Sigma'}$) such that for all nodes $\langle \Sigma'_i, f, i\rangle$ in $\mathbf{G}_{\Sigma'}$ (and corresponding nodes in $\mathbf{G}_{\Sigma}$): $g_{\Sigma'}^{\langle \Sigma'_i, f, i\rangle};\mathbf{Mod}(\sigma) = k_{\Sigma'}^{\langle \Sigma'_i, f, i\rangle} = g_{\Sigma}^{\langle \Sigma'_i, \sigma;f, i\rangle}$.

Lemma 19. $\mathbf{Mod}\colon \mathbf{Sign}^{op} \to \mathbf{Cat}$ *is a functor.*

We then define the transformation $\alpha\colon \mathbf{Mod}_2 \to \Phi;\mathbf{Mod}$, let $\Sigma_2 \in |\mathbf{Sign}_2|$:

$$\alpha(\Sigma_2) = g_{\Phi(\Sigma_2)}^{\langle \Sigma_2, id, 2\rangle}\colon \mathbf{Mod}_2(\Sigma_2) \to \mathbf{Mod}(\Phi(\Sigma_2)).$$

The sentence functor $\mathbf{Sen}$ is defined in similar way; for $\Sigma \in |\mathbf{Sign}|$, $\mathbf{Sen}(\Sigma)$ is a limit of a diagram $\mathbf{E}_{\Sigma} : \mathbf{G}_{\Sigma}^{op} \to \mathbf{Set}$, which is defined similarly as above. Also, $\mathbf{Sen}$ is extended to a functor analogously. The projections on the value of a node $\langle \Sigma_i, f, i\rangle$ in $\mathbf{G}_{\Sigma}^{op}$ will be denoted by $h_{\Sigma}^{\langle \Sigma_i, f, i\rangle} : \mathbf{Sen}(\Sigma) \to \mathbf{Sen}_i(\Sigma_i)$. The transformation $\beta\colon \Phi;\mathbf{Sen} \to \mathbf{Sen}_2$ is defined on $\Sigma_2 \in |\mathbf{Sign}_2|$ as: $\beta(\Sigma_2) = h_{\Phi(\Sigma_2)}^{\langle \Sigma_2, id_{\Phi(\Sigma_2)}, 2\rangle}$.

Lemma 20. *α and β are natural transformations.*

The satisfaction relation in $\mathbf{I}$ is defined, for $\Sigma \in |\mathbf{Sign}|$, $m \in |\mathbf{Mod}(\Sigma)|$ and $\varphi \in \mathbf{Sen}(\Sigma)$, as follows:

$$m \models_{\Sigma} \varphi \iff m_2 \models^2_{\Sigma_2} h_{\Sigma}^{\langle \Sigma_2, f, 2\rangle}(\varphi),$$

where $m_2 \in |\mathbf{Mod}_2(\Sigma_2)|$ is such, that $g_{\Sigma}^{\langle \Sigma_2, f, 2\rangle}(m_2) = m$.

Lemma 21. *The required m_2 always exists, and the definition of the satisfaction relation is independent of the choice of m_2.*

5 Limits and colimits in coINS

Similar results hold for the category **coINS**. The constructions are much like the ones presented above. Like the results on completeness and cocompleteness of **INS** and **coINS**, these theorems have also been known to be true before ([GR02,**?**]), but I have not found explicit constructions. Again, the category **coINS** is not cocomplete, for a reason analogous to **INS** not being cocomplete.

Theorem 22. *The category* **coINS** *is complete.*

Theorem 23. *The category* **scoINS** *is cocomplete.*

6 The categories $\mathbf{INS}_{\mathbf{Sign}}$ and $\mathbf{coINS}_{\mathbf{Sign}}$

Categories of institutions with a fixed signature category exhibit some interesting properties. In particular, as the category of signatures does not change, there is no significant difference between a morphism and a comorphism, and, moreover, the construction of a colimit of a diagram is as easy as the construction of a limit. Also, the constructions of limits and colimits in $\mathbf{INS}_{\mathbf{Sign}}$ and $\mathbf{coINS}_{\mathbf{Sign}}$ can be used to construct limits of diagrams in **INS** and **coINS**.

6.1 Limits and colimits

The construction of an equalizer of two morphisms in $\mathbf{INS}_{\mathbf{Sign}}$ is exactly the same as in Sect. 3.2, as it easily follows from that construction that the signature category of the domain of an equalizer will be equal to $\mathbf{Sign}$.

To construct products in $\mathbf{INS}_{\mathbf{Sign}}$, we need to make a slight change to the construction presented in Sect. 3.1, by making a requirement that the signature category of the product must be $\mathbf{Sign}$, and not $\mathbf{Sign} \times \mathbf{Sign}$. However it's the only change, and the rest of the construction remains the same.

Analogously, we can define products and equalizers in $\mathbf{coINS}_{\mathbf{Sign}}$. Thus, we get:

Theorem 24. *The categories* $\mathbf{INS}_{\mathbf{Sign}}$ *and* $\mathbf{coINS}_{\mathbf{Sign}}$ *are complete.*

Moreover, let's consider an arbitrary morphism $\mu\colon \mathbf{I} \to \mathbf{I}'$ in $\mathbf{INS}_{\mathbf{Sign}}$, where $\mu = \langle id_{\mathbf{Sign}}, \alpha, \beta\rangle$. It is easy to check, that $\rho\colon \mathbf{I}' \to \mathbf{I}$, $\rho = \langle id_{\mathbf{Sign}}, \alpha, \beta\rangle$, is an comorphism in $\mathbf{coINS}_{\mathbf{Sign}}$ (in fact, we can change a morphism into a comorphism using such a technique whenever the functor between signature categories has a left adjoint, see [AF95]). More formally:

Fact 25. $\mu\colon \mathbf{I} \to \mathbf{I}'$, *where* $\mu = \langle id_{\mathbf{Sign}}, \alpha, \beta\rangle$ *is an institution morphism if and only if* $\rho\colon \mathbf{I}' \to \mathbf{I}$, $\rho = \langle id_{\mathbf{Sign}}, \alpha, \beta\rangle$, *is an institution comorphism.*

Corollary 26. $\mathbf{INS}_{\mathbf{Sign}} \cong (\mathbf{coINS}_{\mathbf{Sign}})^{op}$.

It easily follows from Thm. 24 and Cor. 26 that:

Theorem 27. *The categories* $\mathbf{INS}_{\mathbf{Sign}}$ *and* $\mathbf{coINS}_{\mathbf{Sign}}$ *are cocomplete.*

6.2 "Flattening" a diagram in INS to a diagram in coINS

Suppose we have a diagram $\mathbf{D}\colon \mathbf{G} \to \mathbf{INS}$, which has nodes $\mathbf{I}_i$, $\mathbf{I}_j$ for $i, j \in |\mathbf{G}|$, and morphisms $\mu_{k,i,j} : \mathbf{I}_i \to \mathbf{I}_j$, for $k \in K_{i,j}$, where $K_{i,j}$ is a set of indices. For notational convenience, the coordinate k will be omitted.

Let $\mathbf{Sign}$ and morphisms $\Phi_i : \mathbf{Sign} \to \mathbf{Sign}_i$ be a limit of the diagram $\mathbf{D};\mathbf{C}\colon \mathbf{G} \to \mathbf{Cat}$ (see Def. 8).

Given $\mathbf{D}$, we build another diagram $\mathbf{D}'\colon \mathbf{G} \to \mathbf{INS}_{\mathbf{Sign}}$, with nodes $\mathbf{I}'_i$ and morphisms between them $\mu'_{i,j}\colon \mathbf{I}'_i \to \mathbf{I}'_j$.

Each node $\mathbf{I}_i = \langle \mathbf{Sign}_i, \mathbf{Mod}_i, \mathbf{Sen}_i, \models_i\rangle$ in diagram $\mathbf{D}$ we change to a node $\mathbf{I}'_i$ in $\mathbf{D}'$ with the signature category $\mathbf{Sign}$ in the following way:

$$\mathbf{I}'_i = \langle \mathbf{Sign}, \Phi_i^{op};\mathbf{Mod}_i, \Phi_i;\mathbf{Sen}_i, \Phi_i;\models^i\rangle,$$

where $\Phi_i;\models^i$ is a relation that for $\Sigma \in |\mathbf{Sign}|$ is equal to $\models^i_{\Phi_i(\Sigma)}$.

It is easy to check that the satisfaction condition in $\mathbf{I}'_i$ holds.

A morphism $\mu_{i,j} = \langle \Phi_{i,j}, \alpha_{i,j}, \beta_{i,j}\rangle$ in $\mathbf{D}$ is changed to a morphism in $\mathbf{D}'$:

$$\mu'_{i,j} = \langle id, \Phi_i^{op} \cdot \alpha_{i,j}, \Phi_i \cdot \beta_{i,j}\rangle.$$

This definition is correct, as from the construction of **Sign** we have $\Phi_i;\Phi_{i,j} = \Phi_j$, hence:

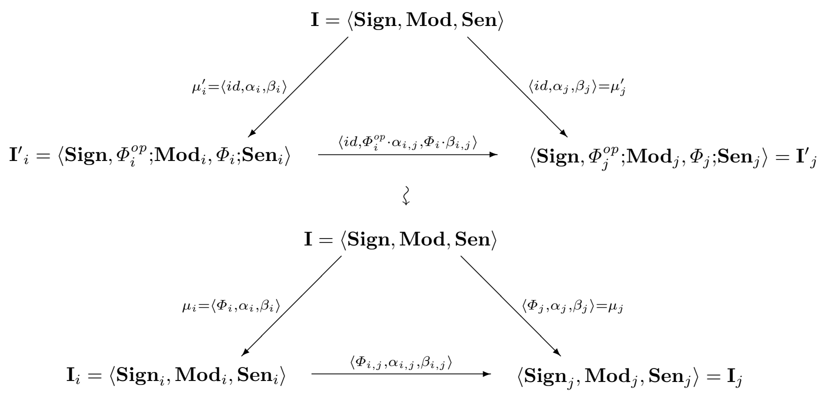
$$\Phi_i^{op} \cdot \alpha_{i,j} \colon \Phi_i^{op};\mathbf{Mod}_i \to \Phi_i^{op};\Phi_{i,j}^{op};\mathbf{Mod}_j = \Phi_j^{op};\mathbf{Mod}_j,$$

and similarly for β. The satisfaction condition for that morphism holds, which follows immediately from the satisfaction condition for $\mu_{i,j}$.

For the diagram $\mathbf{D'} \colon \mathbf{G} \to \mathbf{INS_{Sign}}$ we can construct a limit, as it is described in Sect. 6.1, which will be denoted as $\mathbf{I} = \langle \mathbf{Sign}, \mathbf{Mod}, \mathbf{Sen}, \models \rangle$, where $\mathbf{Mod} \colon \mathbf{Sign}^{op} \to \mathbf{Cat}$, $\mathbf{Sen} \colon \mathbf{Sign} \to \mathbf{Set}$. We also get projections $\mu'_i = \langle id, \alpha_i, \beta_i \rangle \colon \mathbf{I} \to \mathbf{I'}_i$, with natural transformations $\alpha_i \colon \mathbf{Mod} \to \mathbf{Mod'}_i$ and $\beta_i \colon \mathbf{Sen'}_i \to \mathbf{Sen}$.

6.3 Translating a limit of the "flattened" diagram to a limit of the original diagram

Having a limit $\mathbf{I}$ of $\mathbf{D'}$, it is easy to construct a limit of $\mathbf{D}$:



For each node, we get a morphism $\mu_i \colon \mathbf{I} \to \mathbf{I}_i$ by taking a functor projecting **Sign** on $\mathbf{Sign}_i$ and natural transformations from the projections in $\mathbf{D'}$: $\mu_i = \langle \Phi_i, \alpha_i, \beta_i \rangle$. From the definitions of "flattening" a node the natural transformations α_i and β_i are such, that: $\alpha_i \colon \mathbf{Mod} \to \Phi_i^{op};\mathbf{Mod}_i$ and $\beta_i \colon \Phi_i;\mathbf{Sen}_i \to \mathbf{Sen}$. It is easy to check, that μ_i is an institution morphism. It is not quite trivial to verify that $\mathbf{I}$ with projections $\mu_i \colon \mathbf{I} \to \mathbf{I}_i$ is in fact a limit of $\mathbf{D}$.

Theorem 28. *The institution* $\mathbf{I}$ *with projections* $\mu_i \colon \mathbf{I} \to \mathbf{I}_i$ *is a limit of diagram* $\mathbf{D}$.

A construction similar to the one presented above can be found for example in [TBG91, Ch. 4, Lem. 2].

7 Changing morphisms into comorphisms

When examining the definitions of morphisms and comorphisms (2, 3), one can see some duality between the two concepts. It would also be useful to have a way of representing morphisms as comorphisms and vice versa. Also, in specific diagrams morphisms and comorphisms may coexists, and it is easier to reason about a diagram if it has only one type of morphisms. One way, described for example in [Mos02b,Mos06] is to replace a morphism with a span of comorphisms. It is also possible to represent a comorphism as a span of two morphisms. However below we will concentrate on the former, as the category **coINS** appears to be the most suitable for investigating the properties of heterogeneous specifications ([Mos02b]).

7.1 Spans of comorphisms

Suppose we have an institution morphism: $\mu\colon \mathbf{I}_1 \to \mathbf{I}_2 = \langle \Phi, \alpha, \beta \rangle$. We define an "intermediary" institution $\mathbf{I}' = \langle \mathbf{Sign}_1, \Phi^{op};\mathbf{Mod}_2, \Phi;\mathbf{Sen}_2, \Phi;\models^2 \rangle$, which consists of a category of signatures from the first institution, and sentence and model functors from the second institution (here, $\Phi;\models^2$ is a relation, which for $\Sigma \in |\mathbf{Sign}_1|$ is equal to $\models^2_{\Phi(\Sigma)}$).

It is easy to check that this definition is correct. We can also define two morphisms, $\mu_1 : \mathbf{I}_1 \to \mathbf{I}'$ and $\mu_2 : \mathbf{I}' \to \mathbf{I}_2$, where $\mu_1 = \langle id, \alpha, \beta \rangle$ and $\mu_2 = \langle \Phi, id, id \rangle$, which are such that $\mu_1;\mu_2 = \mu$.

Morphism, in which the functor between signature categories is an identity, can be easily changed to comorphism (Fact 25). Moreover, starting with a morphism, in which the natural transformations between model and sentence functors are identities, we can easily build a comorphism: it will consist of exactly the same parts (but with the identity natural transformations considered in the "opposite" direction). The domain or codomain of the morphism doesn't change either. Hence, having a morphism, we can build a span of two comorphisms.

Thus, if we have a morphism: $\mathbf{I}_1 \xrightarrow{\langle \Phi,\alpha,\beta \rangle} \mathbf{I}_2$ we can change it to a pair of morphisms: $\mathbf{I}_1 \xrightarrow{\langle id,\alpha,\beta \rangle} \mathbf{I}' \xrightarrow{\langle \Phi,id,id \rangle} \mathbf{I}_2$ and next to a pair of comorphisms, "reversing" the first, and leaving the second without any changes: $\mathbf{I}_1 \; {}_{co}\!\xleftarrow{\langle id,\alpha,\beta \rangle} \mathbf{I}' \xrightarrow{\langle \Phi,id,id \rangle}_{co} \mathbf{I}_2$. Informally, a span of comorphisms expresses "the same" relation between institutions, as the original morphism.

In a very similar way we can change a comorphism into a span of morphisms.

8 Constructing limits of diagrams with each morphism replaced by a span

Having a way of representing morphisms as spans of comorphisms, it is natural to ask, how do (co)limits of diagrams of institutions correspond to (co)limits of diagrams, in which each morphism has been changed into a span of comorphisms. As the comorphisms used in the spans that replace institution morphisms are quite specific (contain many identities), in the case of limits there exists an easy way to construct them for diagrams obtained in such a way.

Consider a diagram $\mathbf{D}\colon \mathbf{G} \to \mathbf{INS}$ of institutions and their morphisms./

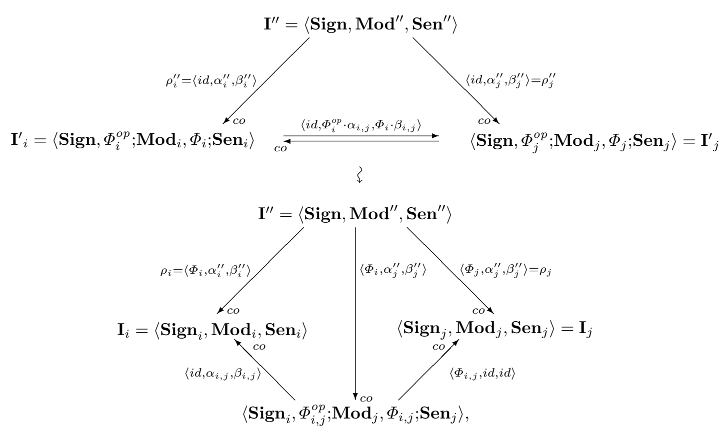
8.1 "Flattened" diagrams and spans

Suppose we "flatten" $\mathbf{D}$ to a diagram $\mathbf{D}'$ as in Sect. 6.2, where also the category **Sign** is defined. We construct new diagrams:

- diagram $\mathbf{coD}\colon \mathbf{coG} \to \mathbf{coINS}$, is a diagram $\mathbf{D}$, in which each morphism has been changed to a span of comorphisms
- diagram $\mathbf{D}''\colon \mathbf{G}^{op} \to \mathbf{coINS}_{\mathbf{Sign}}$ is a diagram $\mathbf{D}'$, in which each morphism has been changed to a comorphism (as in fact 25); its vertices are institutions $\mathbf{I}'_i$ ("flattened" institutions $\mathbf{I}_i$).

The institution that is the vertex of the limit of the diagram $\mathbf{D}''$ will be denoted by $\mathbf{I}'' = \langle \mathbf{Sign}, \mathbf{Mod}'', \mathbf{Sen}'', \models'' \rangle$, with projections $\rho''_i : \mathbf{I}'' \to \mathbf{I}'_i = \langle id, \alpha''_i, \beta''_i \rangle$ for $i \in |\mathbf{G}|$.

From a limit of the diagram $\mathbf{D}''$ we can easily get a limit of the diagram $\mathbf{coD}$:



where we put $\rho_i\colon \mathbf{I}'' \to \mathbf{I}_i = \langle \Phi_i, \alpha''_i, \beta''_i \rangle$, for $i \in |\mathbf{G}|$.

Theorem 29. *The institution and comorphisms constructed above are a limit of the diagram* **coD**.

8.2 Relations between limits of diagrams and limits of diagrams of spans

So, we can construct a limit of a diagram of institutions and institution morphisms, and a limit of a diagram, in which each morphism has been replaced by a span of comorphisms. The natural question is how much the two limits are related. Informally, the limit of the original diagram in **INS** is an institution that is "richer" than all the institutions in the diagram, while the limit of this diagram in **coINS** is "poorer" then all institutions in the diagram. Moreover, an institution morphism "represents" a richer institution in a simpler one, so, if a relation exists, it can be in the form of a morphism from $\mathbf{I}$ to $\mathbf{I}''$ (or a comorphism from $\mathbf{I}''$ to $\mathbf{I}$).

It is the case that such a morphism always exist, and in some diagrams there can be many of them. We can build one morphism from $\mathbf{I}$ to $\mathbf{I}''$ for each node of the diagram, but morphisms built for vertices connected by any path in the graph turn out to be the same. Hence, we can build one morphism for each connected component of the graph. Of course, some of them may turn out to coincide—but only in specific cases.

Fact 30. *For each node $i \in |\mathbf{G}|$, $\langle id, \alpha_i;\alpha_i'', \beta_i'';\beta_i \rangle : \mathbf{I} \to \mathbf{I}''$ is an institution morphism. Moreover, from fact 25, for each node $i \in |\mathbf{G}|$, $\langle id, \alpha_i;\alpha_i'', \beta_i'';\beta_i \rangle : \mathbf{I}'' \to \mathbf{I}$ is an institution comorphism.*

Fact 31. *For vertices $i, j \in |\mathbf{G}|$ connected by any path in the graph we have: $\langle id, \alpha_i;\alpha_i'', \beta_i'';\beta_i \rangle = \langle id, \alpha_j;\alpha_j'', \beta_j'';\beta_j \rangle$.*

9 Conclusions and further work

In this paper, the constructions of limits and colimits in categories of institutions have been presented; the results on completeness and cocompleteness of these categories have been known before, however the proofs did not show direct constructions. Explicit constructions are needed when these theorems are to be apllied to a specific diagram of institutions.

Moreover, as the constructions of limits and colimits turn out to be rather different, it seems that institution morphisms and comorphisms are not dual concepts, as a first intuition may suggest.

The properties of diagrams of institutions with a fixed signature category are also presented, as well as means of translating an arbitrary diagram to a diagram with a fixed signature category, and the relations between the two diagrams. The final part of the article describes some connections between limits of diagrams with morphisms, and limits of diagrams in which each morphism has been changed into a span of comorphisms.

What remains to be investigated, is the possible relation between (co)limits of diagrams of institutions and corresponding Grothendieck institutions, as well as how these results apply to specification theory.

References

[AF95] M. Arrais and José Luiz Fiadeiro. Unifying theories in different institutions. In Haveraaen et al. [HOD96], pages 81–101.

[Dia02] Razvan Diaconescu. Grothendieck institutions. *Applied Categorical Structures*, 10(4):383–402, 2002.

[GB83] Joseph A. Goguen and Rod M. Burstall. Introducing institutions. In Edmund M. Clarke and Dexter Kozen, editors, *Logic of Programs*, volume 164 of *LNCS*, pages 221–256. Springer, 1983.

[GB85] Joseph A. Goguen and Rod M. Burstall. A study in the functions of programming methodology: Specifications, institutions, charters and parchments. In Pitt et al. [PAPR86], pages 313–333.

[GB92] Joseph Goguen and Rod Burstall. Institutions: abstract model theory for specification and programming. *J. ACM*, 39(1):95–146, January 1992.

[GR02] Joseph A. Goguen and Grigore Rosu. Institution morphisms. *Formal Asp. Comput.*, 13(3-5):274–307, 2002.

[HOD96] Magne Haveraaen, Olaf Owe, and Ole-Johan Dahl, editors. *Recent Trends in Data Type Specification, 11th Workshop on Specification of Abstract Data Types Joint with the 8th COMPASS Workshop, Oslo, Norway, September 19-23, 1995, Selected Papers*, volume 1130 of *LNCS*. Springer, 1996.

[Mac71] Saunders MacLane. *Categories for the Working Mathematician*. Number 5 in Graduate Texts in Mathematics. Springer-Verlag, 1971.

[MB99] Wiesaw Pawlowski Marek Bednarczyk, Andrzej Borzyszkowski. Generalized congruences. *Theory and Applications of Categories*, 5(11):266–280, 1999.

[Mos02a] Till Mossakowski. Comorphism-based Grothendieck logics. In Krzysztof Diks and Wojciech Rytter, editors, *MFCS*, volume 2420 of *LNCS*, pages 593–604. Springer, 2002.

[Mos02b] Till Mossakowski. Foundations of heterogeneous specification. In Martin Wirsing, Dirk Pattinson, and Rolf Hennicker, editors, *WADT*, volume 2755 of *LNCS*, pages 359–375. Springer, 2002.

[Mos06] Till Mossakowski. Institutional 2-cells and Grothendieck Institutions. In Kokichi Futatsugi, Jean-Pierre Jouannaud, and José Meseguer, editors, *Essays Dedicated to Joseph A. Goguen*, volume 4060 of *LNCS*, pages 124–149. Springer, 2006.

[PAPR86] David H. Pitt, Samson Abramsky, Axel Poigné, and David E. Rydeheard, editors. *Category Theory and Computer Programming, Tutorial and Workshop, Guildford, UK, September 16-20, 1985 Proceedings*, volume 240 of *LNCS*. Springer, 1986.

[Paw95] Wiesaw Pawlowski. Context institutions. In Haveraaen et al. [HOD96], pages 436–457.

[Ros99] Grigore Rosu. Kan extensions of institutions. *J. UCS*, 5(8):482–493, 1999.

[ST88] Donald Sannella and Andrzej Tarlecki. Specifications in an arbitrary institution. *Inf. Comput.*, 76(2-3):165–210, 1988.

[Tar85] Andrzej Tarlecki. Bits and pieces of the theory of institutions. In Pitt et al. [PAPR86], pages 334–365.

[Tar95] Andrzej Tarlecki. Moving between logical systems. In Haveraaen et al. [HOD96], pages 478–502.

[Tar00] Andrzej Tarlecki. Towards heterogeneous specifications. In D. Gabbay and M. de Rijke, editors, *Frontiers of Combining Systems 2*, Studies in Logic and Computation, pages 337–360. Research Studies Press, 2000.

[TBG91] Andrzej Tarlecki, Rod M. Burstall, and Joseph A. Goguen. Some fundamental algebraic tools for the semantics of computation: Part 3: Indexed categories. *Theor. Comput. Sci.*, 91(2):239–264, 1991.

[War07] Adam Warski. Granice i kogranice w roznych kategoriach instytucji (Limits and colimits in various categories of institutions). Master's thesis, University of Warsaw, 2007. in Polish.